

Finitely generated ideals in Fréchet algebras and a famous problem of Gleason in the theory of SCV

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Abstract. We establish the Gleason result for finitely generated ideals in the context of Fréchet algebras, and, in particular, provide an affirmative answer to the question about the Gleason result in (uniform) commutative Fréchet algebras (posed by Carpenter in 1970). As a welcome bonus of our method, we characterize *locally Stein algebras*, and, as an application of this characterization, we also provide an affirmative answer to the Gleason problem for such algebras, which recaptures all the classical results on the Gleason problem in the theory of several complex variables.

Keywords: Fréchet algebra of power series in k variables, Arens-Michael representation, finitely generated maximal ideal, analytic variety, Stein algebra, Gleason problem, Gleason A -property

1 Introduction and statement of the Main Theorem

An important subject in the theory of Fréchet algebras is the question of the existence of analytic structure in spectra. The detailed study of this problem for uniform Fréchet algebras is discussed in [10], especially the work of Brooks, Carpenter, Goldmann and Kramm. In particular, several function algebraic characterizations of certain types of Stein algebras are given, including a characterization of Stein algebras by intrinsic properties within the class of uniform Fréchet algebras (see [10, Theorem 11.1.4]; Kramm's Theorem); but, as far as we know, there are no characterizations obtained by studying the ideal structure of a Fréchet algebra since 1964. In fact, the problem of determining what kinds of ideals can be found in arbitrary Fréchet algebras arises not only from the aesthetic imperative to understand the internal structure of these algebras but also from certain applications. For example, the proof of the famous Wiener's Theorem is one of the early celebrated accomplishments of the theory of Banach algebras. The Wiener's original proof was a good deal more complicated but the use of Banach algebra technique made it very easy. In this paper, we will experience same

kind of phenomenon again in respect of an affirmative solution of the Gleason problem in the theory of SCV (see 4.4 below). In fact, the solution of the Gleason problem for locally Stein algebras (defined below) recaptures all the classical results on the Gleason problem in the theory of SCV.

We remark that an ideal I of a Fréchet algebra A may be finitely generated in two senses; it may be the least ideal containing a certain finite set F or it may be the least closed ideal containing F . The former situation, called algebraically finitely generated, is rare, and the latter case is common. Finitely generated closed ideals are distinctly uncommon in Fréchet algebras. However, algebraically finitely generated ideals are not trivial, and do form, in a natural way, an analytic variety provided that they belong to the spectrum of A . Also, very rarely is it possible to give an analytic structure to the whole spectrum and therefore it is of interest to know conditions which ensure that parts of the spectrum of a Fréchet algebra can be given an analytic structure. In [12], Loy gave a sufficient condition for the existence of local analytic structure in spectra of certain commutative Fréchet algebras.

This paper contains a continuation of the work begun in [15], [16] and [17], and we shall feel free to use the terminology and conventions established there. However, this current work is specifically concerned with the deter-

mination of sufficient conditions for the existence of local analytic structure in the spectrum of a commutative Fréchet algebra by studying the structure of the algebra. (See Main Theorem below.) As a consequence, we characterize locally Stein algebras by intrinsic properties within the class of Fréchet algebras. Also we extend the result of Gleason to finitely generated ideals in Fréchet algebras, answering affirmatively the question posed in [5] for Fréchet algebras. Though the present paper is primarily addressed to functional analysts; we hope that complex analysts may also find the sufficient conditions interesting from an applications point of view, for example, see Corollary 4.1 and 4.4 below, and references (and their reviews, too) to [9] in MathSciNet (MR0159241 (28 #2458)).

Throughout the paper, “algebra” will mean a non-zero, complex commutative algebra with identity e . We recall that a *Fréchet algebra* is a complete, metrizable locally convex algebra A whose topology τ may be defined by an increasing sequence $(p_m)_{m \geq 1}$ of submultiplicative seminorms. The basic theory of Fréchet algebras was introduced in [15]. The principal tool for studying Fréchet algebras is the Arens-Michael representation, in which A is given by an inverse limit of Banach algebras A_m , given below. A Fréchet algebra A is called a *uniform Fréchet algebra* if for each $m \geq 1$ and for each

$$x \in A, p_m(x^2) = p_m(x)^2.$$

Let A be a Fréchet algebra, with its topology defined by an increasing sequence $(p_m)_{m \geq 1}$ of submultiplicative seminorms. For each m , let $Q_m : A \rightarrow A/\ker p_m$ be the quotient map. Then $A/\ker p_m$ is naturally a normed algebra, normed by setting $\|x + \ker p_m\|_m = p_m(x)$ ($x \in A$). We let $(A_m; \|\cdot\|_m)$ be the completion of $A/\ker p_m$; henceforth we consider Q_m as a mapping from A into A_m . Then $d_m(x + \ker p_{m+1}) = x + \ker p_m$ ($x \in A$) extends to a norm decreasing homomorphism $d_m : A_{m+1} \rightarrow A_m$ such that

$$A_1 \xleftarrow{d_1} A_2 \xleftarrow{d_2} A_3 \longleftarrow \cdots \longleftarrow A_m \xleftarrow{d_m} A_{m+1} \longleftarrow \cdots$$

is an inverse limit sequence of Banach algebras; and bicontinuously $A = \varprojlim (A_m; d_m)$. This is called an *Arens-Michael representation* of A . For an element $x \in A$, we may write $x_m = Q_m(x)$; it is then evident that, for each $x \in A$, the sequence (x_m) is an element of $\varprojlim (A_m; d_m)$.

We note that: (i) there is a one-to-one correspondence $\phi \rightarrow \ker \phi$ between the set $M(A)$ of all non-zero continuous complex homomorphisms on A and the closed maximal ideals in A , and (ii) maximal ideals of Fréchet algebras are not, in general, closed (see [10, Example, p. 83]).

Let $k \in \mathbb{N}$. We write \mathcal{F}_k for the algebra $\mathcal{C}[[X_1, X_2, \dots, X_k]]$ of all formal power series in k commuting indeterminates X_1, X_2, \dots, X_k , with complex

coefficients. A fuller description of this algebra is given in [6, §1.6], and for the algebraic theory of \mathcal{F}_k , see [21, Chapter VII]; we briefly recall some notation, which will be used throughout the paper. Let $k \in \mathbb{N}$, and let $J = (j_1, j_2, \dots, j_k) \in \mathbb{Z}^{+k}$. Set $|J| = j_1 + j_2 + \dots + j_k$; ordering and addition in \mathbb{Z}^{+k} will always be component-wise. A generic element of \mathcal{F}_k is denoted by

$$\sum_{J \in \mathbb{Z}^{+k}} \lambda_J X^J = \sum \{ \lambda_{(j_1, j_2, \dots, j_k)} X_1^{j_1} X_2^{j_2} \dots X_k^{j_k} : (j_1, j_2, \dots, j_k) \in \mathbb{Z}^{+k} \}.$$

The algebra \mathcal{F}_k is a Fréchet algebra when endowed with the weak topology τ_c defined by the projections $\pi_I : \mathcal{F}_k \rightarrow \mathbb{C}, I \in \mathbb{Z}^{+k}$, where $\pi_I(\sum_{J \in \mathbb{Z}^{+k}} \lambda_J X^J) = \lambda_I$. A defining sequence of seminorms for \mathcal{F}_k is (p'_m) , where $p'_m(\sum_{J \in \mathbb{Z}^{+k}} \lambda_J X^J) = \sum_{|J| \leq m} |\lambda_J|$ ($m \in \mathbb{N}$). A *Fréchet algebra of power series in k commuting indeterminates* is a subalgebra A of \mathcal{F}_k such that A is a Fréchet algebra containing the indeterminates X_1, X_2, \dots, X_k and such that the inclusion map $A \hookrightarrow \mathcal{F}_k$ is continuous (equivalently, the projections $\pi_I, I \in \mathbb{Z}^{+k}$, are continuous linear functionals on A) [7]. When it can cause no confusion, we may use the term “algebra of power series in \mathcal{F}_k ” for “algebra of power series in k commuting indeterminates” in the following; thus, Fréchet algebras of power series in \mathcal{F}_1 are the usual Fréchet algebras of power series. Though Fréchet algebras of power series in \mathcal{F}_k (shortly: FrAPS in \mathcal{F}_k) have been considered earlier by Loy [12], recently these algebras—

and more generally, the power series ideas in general Fréchet algebras—have acquired significance in understanding the structure of Fréchet algebras [1, 4, 7, 15, 16, 17]. For examples of Fréchet algebras of power series, we refer to [4]; also, analogous examples of Fréchet algebras of power series in \mathcal{F}_k can be constructed from the examples given in [4]. In particular, we shall consider *Beurling-Fréchet algebras* $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of *semiweight type* in the following; we define these algebras as follows.

First, recall that ω is a *proper semiweight* if $\omega(N_0) = 0$ for some $N_0 \in \mathbb{N}^k$.

Let $k \in \mathbb{N}$, and let

$$\ell^1(\mathbb{Z}^{+k}, \Omega) := \{f = \sum_{J \in \mathbb{Z}^{+k}} \lambda_J X^J \in \mathcal{F}_k : \sum_{J \in \mathbb{Z}^{+k}} |\lambda_J| \omega_m(J) < \infty \text{ for all } m\},$$

where $\Omega = (\omega_m)$ is a separating and increasing sequence of proper semiweights on \mathbb{Z}^{+k} defined by $\omega_m(N) = p_m(X^N)$. Then $\rho = 0$ if and only if $\ell^1(\mathbb{Z}^{+k}, \Omega)$ is a local Fréchet algebra if and only if the completion of $\ell^1(\mathbb{Z}^{+k}, \omega_m)/\ker p_m$ under the induced norm p_m is a local Banach algebra for all m . So $\ell^1(\mathbb{Z}^{+k}, \Omega)$ is either \mathcal{F}_k or a local FrAPS in \mathcal{F}_k . We call such a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ an *algebra of semiweight type*. We note that the unique maximal ideal of $\ell^1(\mathbb{Z}^{+k}, \Omega)$ is

$$\{f = \sum_{J \in \mathbb{Z}^{+k}} \lambda_J X^J \in \ell^1(\mathbb{Z}^{+k}, \Omega) : \lambda_0 = 0\}.$$

We recall that the spectrum $M(A)$ (with the Gel'fand topology) has an *analytic variety* at $\phi \in M(A)$ if there is a subvariety D containing 0 of a domain in some \mathcal{C}^k and a continuous injection $f : D \rightarrow M(A)$ such that $f(0) = \phi$ and $\hat{x} \circ f \in \text{Hol}(D)$ for all $x \in A$. A uniform Fréchet algebra A is called a *Stein algebra* if it is topologically and algebraically isomorphic to the Fréchet algebra $\text{Hol}(X)$ of all holomorphic functions on some (reduced) Stein space X (see [10, 11.1.1]). We call a Fréchet algebra A a *locally Stein algebra* if a non-empty part of $M(A)$ can be given the structure of a (reduced) Stein space in such a way that the completion in the compact open topology of the algebra of Gel'fand transforms of elements of A , restricted to this part, is the Fréchet algebra of all holomorphic functions on this Stein space.

Let $x \in A$ and let R_x denote the linear operator $A \rightarrow A$ of multiplication by x . A non-zero element $x \in A$ is called a *strong topological divisor of zero* in A if R_x is not an isomorphism into, i.e. a linear homeomorphism of A onto Ax [14, Definition 11.1]. A non-zero element $x \in A$ is a *topological divisor of zero* in A if, whenever a sequence (p_m) of seminorms defines the Fréchet topology of A , there exists $l \in \mathbb{N}$ such that x_l is a topological divisor of 0 in A_l [14, Definition 11.2]. We remark that the two notions of topological divisor of zero agree for normed algebras.

We now state the main result on analytic structure to be proved in this paper. We remark that every algebraically finitely generated, maximal ideal is a closed maximal ideal in A (see [2]). For each $n \geq 1$, M^n is the ideal generated by products of n elements in M .

MAIN THEOREM. *Let A be a commutative, unital Fréchet algebra, with its topology defined by a sequence (p_m) of norms and with the corresponding Arens-Michael isomorphism $A = \varprojlim (A_m; d_m)$. Suppose that A has a maximal ideal M that is algebraically finitely generated, say by t_1, t_2, \dots, t_k , that for each n the homogeneous monomials of degree n in t_1, t_2, \dots, t_k are representatives of a basis for $\overline{M^n}/\overline{M^{n+1}}$; and that the generators t_i , $i = 1, 2, \dots, k$, have the property that t_{i_m} is not a topological divisor of zero in A_m for all sufficiently large m . Then:*

- (i) $A/\bigcap_{n \geq 1} \overline{M^n}$ is a semisimple Fréchet algebra of power series in \mathcal{F}_k ;
- (ii) there is an analytic variety at ϕ , where $M = \ker \phi$;
- (iii) for each $x \in A$, \hat{x} vanishes on a neighborhood of ϕ provided that $x \in \bigcap_{n \geq 1} \overline{M^n}$.

The paper ends with some remarks on the hypotheses of the main theorem. We provide a characterization of locally Stein algebras as a corollary

to the main theorem, and, as an application, we give an affirmative solution to the Gleason problem (see 4.4 below) for locally Stein algebras.

2 Fréchet algebras of power series in \mathcal{F}_k

The proof of the main theorem, presented in Section 3, is broken up into several technical results of some independent interest.

Let M be a closed ideal of a Fréchet algebra A . Then $\overline{M^n}$ for each $n \geq 1$ and $\bigcap_{n \geq 1} \overline{M^n}$ are also closed ideals of A . We now state our two vital technical lemmas (see [15, p. 127]), recalling the Arens-Michael representations of M , $\overline{M^n}$ for each $n \geq 1$ and $\bigcap_{n \geq 1} \overline{M^n}$, and their quotient Fréchet algebras $A/\overline{M^n}$ for each $n \geq 1$ and $A/\bigcap_{n \geq 1} \overline{M^n}$.

Lemma 2.1 *Let M be a closed ideal of A . Then the Arens-Michael isomorphism $A \cong \varprojlim (A_m; d_m)$ induces isomorphisms:*

- (i) $M \cong \varprojlim (M_m; \overline{d_m})$;
- (ii) $\overline{M^n} \cong \varprojlim (\overline{M_m^n}; \overline{d_m})$ ($n \geq 1$);
- (iii) $\bigcap_{n \geq 1} \overline{M^n} \cong \varprojlim (\bigcap_{n \geq 1} \overline{M_m^n}; \overline{d_m})$.

(Here $\overline{d_m} = d_m|_{I_{m+1}}: I_{m+1} \rightarrow I_m$, where $I_m = \overline{Q_m(I)}$ (closure in A_m), whenever I is a closed ideal in A .)

Lemma 2.2 *With the above notation, the Arens-Michael isomorphism $A \cong \varprojlim (A_m; d_m)$ induces isomorphisms:*

$$(i) \quad A/\overline{M^n} \cong \varprojlim (A_m/\overline{M_m^n}; \tilde{d}_m) \quad (n \geq 1);$$

$$(ii) \quad A/\bigcap_{n \geq 1} \overline{M^n} \cong \varprojlim (A_m/\bigcap_{n \geq 1} \overline{M_m^n}; \tilde{d}_m).$$

(Here $\tilde{d}_m : A_{m+1}/\overline{M_{m+1}^n} \rightarrow A_m/\overline{M_m^n}$ is the homomorphism induced by d_m .)

Let M be a closed maximal ideal of a Fréchet algebra A . We shall suppose from now on that $\dim(M/\overline{M^2}) = k$ is finite (it is easy to see that for finitely generated Fréchet algebras this condition is automatically satisfied; see [18, Proposition 2.2] for the Banach case). Then, by the remark following Theorem 2.3 of [18], for each $n \in \mathbb{N}$ the homogeneous monomials of degree n in $t_1, t_2, \dots, t_k \in M$ are representatives of a basis for $\overline{M^n}/\overline{M^{n+1}}$ if and only if $\dim(\overline{M^n}/\overline{M^{n+1}}) = C_{n+k-1, n}$ for all n , and thus M is not nilpotent. This situation arises for separable Fréchet algebras of power series in \mathcal{F}_k (see Theorem 3.1 of [17]), and, in particular, for the uniform closure of the polynomials on the (closed or open) unit poly-disc in \mathcal{O}^k . Thus, in a special case, we have the following

Proposition 2.3 *Let $(A, (p_m))$ be a commutative, unital Fréchet algebra with the Arens-Michael isomorphism $A \cong \varprojlim (A_m; d_m)$, and let M be a closed maximal ideal of A such that: (i) $\bigcap_{n \geq 1} \overline{M^n} = \{0\}$ and (ii) $\dim(\overline{M^n}/\overline{M^{n+1}}) = C_{n+k-1,n}$ for all n . Then there exist $t_1, t_2, \dots, t_k \in M$ such that $\overline{M^n} = \overline{M^{n+1}} \oplus \text{sp}\{t^I : |I| = n\}$ for each $n \geq 1$. Assume further that each p_m is a norm. Then, for each sufficiently large m , M_m is a non-nilpotent maximal ideal of A_m such that: (a) $\bigcap_{n \geq 1} \overline{M_m^n} = \{0\}$ and (b) $\dim(\overline{M_m^n}/\overline{M_m^{n+1}}) = C_{n+k-1,n}$ for all n .*

Proof. The first half of the proof has already been discussed above. Assume further that each p_m is a norm. Then it is clear that M_m is not nilpotent for each m since M is not nilpotent, and so, by Lemma 2.1 (ii), we have $\overline{Q_m(M)^n} = \overline{M_m^n} \neq \{0\}$ for all n, m . Also, since $\bigcap_{n \geq 1} \overline{M^n} = \{0\}$, we have $\bigcap_{n \geq 1} \overline{M_m^n} = \{0\}$ for each m , by Lemma 2.1 (iii) and [6, Corollary A.1.25]. Since M is a closed maximal ideal of A , we have that $A = M + \mathcal{C}$. Thus $Q_m(M) + \mathcal{C} = M + \mathcal{C}$ is dense in A_m , and so also is $M_m + \mathcal{C}$. Since M_m is closed in A_m , we have that $A_m = M_m + \mathcal{C}$. As it is not true that $M_m = A_m$ for infinitely many $m \in \mathbb{N}$, this proves that M_m is a maximal ideal of A_m for each sufficiently large m . Repeating this argument for $M = \overline{M^2} \oplus \text{sp}\{t_j : j = 1, 2, \dots, k\}$ and using the fact that

$\overline{Q_m(\overline{M^2})} = \overline{M_m^2}$, we have $\dim(M_m/\overline{M_m^2}) = k$ for each sufficiently large m .

Since, for each sufficiently large m , M_m is a non-nilpotent maximal ideal of A_m such that $\bigcap_{n \geq 1} \overline{M_m^n} = \{0\}$ and $\dim(M_m/\overline{M_m^2}) = k$, for such m we also obtain $\dim(\overline{M_m^n}/\overline{M_m^{n+1}}) = C_{n+k-1,n}$ for all n , by $\dim(\overline{M^n}/\overline{M^{n+1}}) = C_{n+k-1,n}$ for all n and by the consequence of [19, Theorem 1]. \square

We remark that, in the case where $\dim(M/\overline{M^2}) = 1$, one deduces $\dim(\overline{M^n}/\overline{M^{n+1}}) = 1$ for all n in [15, Proposition 2.3], and so we do not require a stronger hypothesis on $\dim(\overline{M^n}/\overline{M^{n+1}})$, but then we do require M to be non-nilpotent there. We have an easy counter-example (see [17]) to show that a stronger hypothesis on $\dim(\overline{M^n}/\overline{M^{n+1}})$ is necessary.

Let $k > 1$. Let A be a Fréchet algebra of power series in \mathcal{F}_k . Then A is an integral domain. Set $M = \ker \pi_0 = \{\sum_{J \in \mathbb{Z}^{+k}} \lambda_J X^J \in A : \lambda_0 = 0\}$ (where 0 as a suffix denotes $\{0, 0, \dots, 0\}$). Then M is a non-nilpotent, closed maximal ideal of A . Note that π_0 is a continuous projection on A , which is also a complex homomorphism on A . Further, $\overline{M^n} \subset \ker \pi_{N-1}$ for each $N \in \mathbb{N}^n$, where $M^n = \sum_{|N|=n} X^N A$, so that $\bigcap_{n \geq 1} \overline{M^n} = \{0\}$. The following result generalizes this argument. We recall that a Fréchet algebra of power series in \mathcal{F}_k satisfies *condition (E)* if there is a sequence (γ_K) of positive reals such that $(\gamma_K^{-1} \pi_K)$ is an equicontinuous family [12],

and, by [17, Theorem 3.10], Fréchet algebras of power series in \mathcal{F}_k (except the Beurling-Fréchet algebras $\ell^1(\mathbf{Z}^{+k}, \Omega)$ of semi-weight type) satisfy this condition (see [17] for details on the algebra $\ell^1(\mathbf{Z}^{+k}, \Omega)$).

Theorem 2.4 *Let A be a Fréchet algebra, $\theta : A \rightarrow B$ a homomorphism of A onto a Fréchet algebra of power series B in \mathcal{F}_k such that B is not equal to a Beurling-Fréchet algebra $\ell^1(\mathbf{Z}^{+k}, \Omega)$ of semi-weight type. Then A contains a non-nilpotent, closed maximal ideal M such that $\bigcap_{n \geq 1} \overline{M^n} = \ker \theta$.*

Remark. We first note that the range of θ is not one-dimensional, so, by [17, Theorem 4.1], θ is continuous whenever B is not equal to the Beurling-Fréchet algebra $\ell^1(\mathbf{Z}^{+k}, \Omega)$ of semi-weight type, and hence one can follow the proof given in [13, Theorem 1] for the Banach algebra case. In view of the further remarks on this theorem (cf. [16, REMARK]), it is of interest to construct counterexamples of Fréchet algebras which have discontinuous epimorphisms onto the Beurling-Fréchet algebra $\ell^1(\mathbf{Z}^{+k}, \Omega)$ of semi-weight type (and, in particular, onto \mathcal{F}_k); cf. [6, Theorem 5.5.19], and [20, §2] in which Thomas provided necessary conditions for the existence of an epimorphism from a Fréchet algebra onto \mathcal{F} .

If we delete the hypothesis that $\bigcap_{n \geq 1} \overline{M^n} = \{0\}$ from Theorem 3.1 of [17], then we have the following theorem; we will merely sketch a proof.

Theorem 2.5 *Let A be a commutative, unital Fréchet algebra. Suppose that A has a closed maximal ideal M such that $\dim(\overline{M^n}/\overline{M^{n+1}}) = C_{n+k-1,n}$ for each n . Then $A/\bigcap_{n \geq 1} \overline{M^n}$ is a Fréchet algebra of power series in \mathcal{F}_k .*

Proof. Supposing A satisfies the stated condition, there exist $t_1, t_2, \dots, t_k \in M$ such that $\overline{M^n} = \overline{M^{n+1}} \oplus \text{sp}\{t^I : |I| = n\}$ for each $n \geq 1$, by Proposition 2.3. Let $x \in A$. Then a simple induction on n shows that for $n \geq 1$,

$$x = \sum_{|I| \leq n} \lambda_I t^I + y_n,$$

where $y_n \in \overline{M^{n+1}}$ and the (λ_I) are uniquely determined (in fact, by [18, p. 237], x has a unique partial sum of degree n for each n since $\dim(\overline{M^n}/\overline{M^{n+1}}) = C_{n+k-1,n}$ for all n). Hence the functionals $\pi_J : x \mapsto \lambda_J$ are uniquely defined, and linear for all $J \in \mathbb{N}^k$. Thus we have a homomorphism $\Psi : x \mapsto \sum_{I \in \mathbb{Z}^{+k}} \pi_I(x) t^I$ from A onto an algebra of formal power series in \mathcal{F}_k with kernel $\bigcap_{|I| \geq 0} \ker \pi_I = \bigcap_{n \geq 1} \overline{M^n}$. The inclusion $\bigcap_{|I| \geq 0} \ker \pi_I \subseteq \bigcap_{n \geq 1} \overline{M^n}$ is clear. For the reverse, suppose that $x \in \bigcap_{n \geq 1} \overline{M^n}$ and that x does not belong to $\bigcap_{|I| \geq 0} \ker \pi_I$. Let k be the least index, if there is one, such that $\pi_{|I|=k}(x) \neq 0$. Then $x = \sum_{|I|=k} \pi_I(x) t^I + y_k$, where $y_k \in \overline{M^{k+1}}$. So $t^I_{|I|=k} \in \overline{M^{k+1}}$, a contradiction.

For $x \in A$, let \bar{x} denote the coset $x + \bigcap_{n \geq 1} \overline{M^n}$. Then the mapping $\bar{x} \mapsto \sum_{I \in \mathbb{Z}^{+k}} \pi_I(\bar{x}) t^I$ is an isomorphism from $A/\bigcap_{n \geq 1} \overline{M^n}$ onto an algebra

of formal power series in \mathcal{F}_k . One can now follow proof of Theorem 3.1 of [15], in order to establish the theorem. \square

As a corollary, we have the following result, with the Beurling-Fréchet algebras $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semi-weight type (including \mathcal{F}_k) as trivial examples. We note that the polynomials in k variables are dense in $\ell^1(\mathbb{Z}^{+k}, \Omega)$, and so, by [17, Theorem 3.1], $M = \ker \pi_0$ is a non-nilpotent, closed maximal ideal such that $\bigcap_{n \geq 1} \overline{M^n} = \{0\}$ and $\dim(\overline{M^n}/\overline{M^{n+1}}) = C_{n+k-1,n}$ for all n ; the Beurling-Fréchet algebras $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semi-weight type (including \mathcal{F}_k) do not satisfy the latter half of Proposition 2.3 since the topology of $\ell^1(\mathbb{Z}^{+k}, \Omega)$ is defined by proper seminorms.

Corollary 2.6 *Let A be a commutative, unital Fréchet algebra. Suppose that the polynomials in e and t_1, t_2, \dots, t_k are dense in A , and that $\dim(\overline{M^n}/\overline{M^{n+1}}) = C_{n+k-1,n}$ for each n . Then $A/\bigcap_{n \geq 1} \overline{M^n}$ is a Fréchet algebra of power series in \mathcal{F}_k .*

3 Proof of the Main Theorem

First, we prove a Banach algebra analogue of the main theorem in the following lemma; the method of proof of the lemma will be used again in the

proof of the main theorem.

Lemma 3.1 *Let A be a Banach algebra which has an algebraically finitely generated maximal ideal M generated by t_1, t_2, \dots, t_k for some $k \in \mathbb{N}$. Suppose that $\dim(\overline{M^n}/\overline{M^{n+1}}) = C_{n+k-1,n}$ for all n . Then:*

- (i) $A/\bigcap_{n \geq 1} \overline{M^n}$ is a semisimple Banach algebra of power series in \mathcal{F}_k ;
- (ii) there is an analytic variety at ϕ , where $M = \ker \phi$;
- (iii) for each $x \in A$, \hat{x} vanishes on a neighbourhood of ϕ provided that $x \in \bigcap_{n \geq 1} \overline{M^n}$.

Proof. First, we note that $\overline{M^n} = \overline{M^{n+1}} \oplus \text{sp}\{t^I : |I| = n\}$ for each $n \in \mathbb{N}$, and that $\bigcap_{n \geq 1} \overline{M^n}$ is a closed ideal of A . By Theorem 2.5, $B = A/\bigcap_{n \geq 1} \overline{M^n}$ is a Banach algebra of power series in \mathcal{F}_k . For $x \in A$, let \bar{x} denote the coset $x + \bigcap_{n \geq 1} \overline{M^n}$ (which is, in fact, $\sum_{I \in \mathbb{Z}^{+k}} \pi_I(\bar{x})t^I$). Since B is an integral domain, $\bar{t}_1, \bar{t}_2, \dots, \bar{t}_k$ are certainly not zero divisors. Also the image of M under the quotient map is an algebraically finitely generated, maximal ideal generated by $\bar{t}_1, \bar{t}_2, \dots, \bar{t}_k$, i.e., $M/\bigcap_{n \geq 1} \overline{M^n} = \sum_{i=1}^k B\bar{t}_i$, and so, by [2], it is closed. Since \bar{t}_i , $i = 1, 2, \dots, k$, are in $M/\bigcap_{n \geq 1} \overline{M^n}$, they do not belong to $\text{Inv } B$, and so each $B\bar{t}_i$ is an ideal in B such that $\bar{t}_i \in B\bar{t}_i$. In fact, each $B\bar{t}_i$ is a closed, principal ideal in B since it cannot

be dense as it is contained in a closed maximal ideal $M/\bigcap_{n \geq 1} \overline{M^n}$. Hence each \bar{t}_i is not a topological divisor of zero in B . Thus, for each i , the map $R_{\bar{t}_i} : \bar{x} \mapsto \bar{x}\bar{t}_i$ has a continuous inverse, and a simple induction gives $\|\pi_I\| \leq (2\|R_{\bar{t}_i}^{-1}\|)^{|I|}$ for $|I| \in \mathbb{Z}^+$. If Ω is an open poly-disc centered at 0 and radius $\delta < \min_i ((2\|R_{\bar{t}_i}^{-1}\|)^{-1})$, then the map $\bar{x} \mapsto \sum_{I \in \mathbb{Z}^{+k}} \pi_I(\bar{x}) z^I$ of B into $H(\bar{\Omega})$ (the poly-disc algebra) is an algebraic isomorphism so that B must be semisimple. This proves (i).

For (ii), define functionals $\{\phi_\lambda|A : \|\lambda\|_{\mathcal{C}^k} < \delta\}$ on A by $\phi_\lambda|A : x \mapsto \sum_{I \in \mathbb{Z}^{+k}} \pi_I(\bar{x}_1) \lambda^I$. Then $\Gamma : \lambda \mapsto \phi_{\lambda\delta}|A$ is easily seen to be an analytic variety at ϕ .

To prove (iii), let $B' = A \oplus A \oplus \cdots \oplus A \oplus \mathcal{C}$, where A is repeated k times, and set

$$\|\oplus_{i=1}^k x_i \oplus \alpha\| = \max\{\|x_1\|, \|x_2\|, \dots, \|x_k\|, \|\alpha\|\}.$$

Then B' is a Banach algebra; the algebraic operations are coordinatewise. For each $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathcal{C}^k$, define linear operators on A by $T_{\lambda_i} : x \mapsto (t_i - \lambda_i e)x$, and a linear map on B' by $T_\lambda := \oplus_{i=1}^k T_{\lambda_i}$. Since M is an algebraically finitely generated, maximal ideal of A with generators t_1, t_2, \dots, t_k , clearly $T_0 : \oplus_{i=1}^k x_i \mapsto \sum_{i=1}^k t_i x_i$ is a semi-Fredholm operator from B' onto A , with deficiency 1, and so there is $\eta > 0$ such that

$T_\lambda = T_0 - \lambda e$ has deficiency ≤ 1 for $\|\lambda\|_{\mathcal{C}^k} < \eta$. Let $\epsilon = \min(\delta, \eta)$. Then if $\|\lambda\|_{\mathcal{C}^k} < \epsilon$, $T_\lambda(B') \subset \ker \phi_\lambda$ and $\text{codim } T_\lambda(B') \geq \text{codim } \ker \phi_\lambda$. So $\ker \phi_\lambda = T_\lambda(B')$.

Let $\Delta_1 := \{z \in \mathcal{C}^k : \|z\|_{\mathcal{C}^k} < \epsilon\delta^{-1}\}$, $U := \{\psi \in M(A) : |\psi(t_i)| < \epsilon, i = 1, 2, \dots, k\}$. Then from what we have just shown, $\Gamma : \Delta_1 \rightarrow U$ is a continuous bijection. So if $x \in \bigcap_{n \geq 1} \overline{M^n}$, then clearly $\hat{x}(\psi_{\lambda\delta}) = \sum_{I \in \mathcal{I}^{+k}} \pi_I(\bar{x}_1)(\lambda\delta)^I = 0$ for all $\psi_{\lambda\delta} \in U$. \square

We now give proof of the main theorem.

Proof of main theorem. First, we note that

$$\overline{M^n} = \overline{M^{n+1}} \oplus \text{sp}\{t^I : |I| = n\} \supset \overline{M^{n+1}}$$

properly for each $n \in \mathbb{N}$, and so $\bigcap_{n \geq 1} \overline{M^n}$ is a closed ideal of A . We have the following conclusions:

(a) $M \cong \varprojlim M_m$; $\overline{M^n} \cong \varprojlim \overline{M_m^n}$; $\bigcap_{n \geq 1} \overline{M^n} \cong \varprojlim \bigcap_{n \geq 1} \overline{M_m^n}$ by Lemma 2.1.
(b) $A/\overline{M^n} \cong \varprojlim A_m/\overline{M_m^n}$; $A/\bigcap_{n \geq 1} \overline{M^n} \cong \varprojlim A_m/\bigcap_{n \geq 1} \overline{M_m^n}$; $\overline{M^n}/\overline{M^{n+1}} \cong \varprojlim \overline{M_m^n}/\overline{M_m^{n+1}}$ by Lemma 2.2. We have the last Arens-Michael isomorphism as the ideals $\overline{M^n}$ are all distinct.

(c) By Theorem 2.5, $B = A/\bigcap_{n \geq 1} \overline{M^n}$ is a Fréchet algebra of power series in \mathcal{F}_k .

(d) We first recall that each p_m is a norm. Then, by Proposition 2.3, M_m is a maximal ideal in A_m for sufficiently large m such that $\dim(\overline{M_m^n}/\overline{M_m^{n+1}}) = C_{n+k-1,n}$ for each n since $\dim(\overline{M^n}/\overline{M^{n+1}}) = C_{n+k-1,n}$ for each n . So, by Theorem 2.5, $B_m = A_m/\bigcap_{n \geq 1} \overline{M_m^n}$ is a Banach algebra of power series in \mathcal{F}_k for sufficiently large m . Hence, by passing to a suitable subsequence of (q_m) defining the same Fréchet topology of B , we conclude, without loss of generality, that each B_m is a Banach algebra of power series in \mathcal{F}_k . Thus, by [17, Theorem 3.10], B is not equal to a Beurling-Fréchet algebra $\ell^1(\mathcal{Z}^{+k}, \Omega)$ of semi-weight type, and, by [17, Theorem 3.7], the topology of B is, indeed, defined by the sequence (q_m) of norms. Not only that, but, by [17, Corollary 4.3], B has a unique topology as a Fréchet algebra so that each q_m can be taken as a quotient norm induced by the norm p_m .

Next, following the arguments given in the proof of Lemma 3.1, $\bar{t}_1, \bar{t}_2, \dots, \bar{t}_k$ are certainly not zero divisors. Also, by [2], $M/\bigcap_{n \geq 1} \overline{M^n} = \sum_{i=1}^k B\bar{t}_i$, is closed. Since \bar{t}_i , $i = 1, 2, \dots, k$, are in $M/\bigcap_{n \geq 1} \overline{M^n}$, they do not belong to $\text{Inv } B$ by [14, Theorem 5.4] (in the unital case), and so each $B\bar{t}_i$ is an ideal in B such that $\bar{t}_i \in B\bar{t}_i$. In fact, each $B\bar{t}_i$ is a closed, principal ideal in B since it cannot be dense as it is contained in a closed maximal ideal $M/\bigcap_{n \geq 1} \overline{M^n}$. So, for each i , the mapping $R_{\bar{t}_i} : \bar{x} \mapsto \bar{x}\bar{t}_i$ of B onto $B\bar{t}_i$

is a homeomorphism, by the open mapping theorem. Now, for each m and i , the mapping $(R_{\bar{t}_i})_m : \bar{x} \mapsto \bar{x}\bar{t}_i$ of (B, q_m) into $(B\bar{t}_i, q_m)$ is a continuous linear transformation, being a multiplication operator on the normed algebra (B, q_m) . Lifting to the completions, for each m and i , $R_{\bar{t}_{i_m}} : \bar{x}_m \mapsto \bar{x}_m\bar{t}_{i_m}$ of B_m into $(B\bar{t}_i)_m$ is continuous.

Assuming that for some $k \in \mathbb{N}$ the generators $\bar{t}_1, \bar{t}_2, \dots, \bar{t}_k$ have the property that t_{i_m} is not a topological divisor of zero in A_m for all sufficiently large m , say $m \geq n$, we have t_{i_m} is not a zero divisor, and $A_m t_{i_m}$ is a closed ideal in A_m containing At_i so that $A_m t_{i_m} = (At_i)_m$ for all $m \geq n$. Hence $B_m \bar{t}_{i_m} = (B\bar{t}_i)_m$ for all $m \geq n$, and so $R_{\bar{t}_{i_m}}$ is surjective. In fact, it is a homeomorphism, by the open mapping theorem. Thus, without loss of generality, each $R_{\bar{t}_{i_m}}$ has continuous inverse. In particular, by Lemma 3.1, we have

$$\| \pi_I^{(1)} \| \leq (2 \| R_{\bar{t}_{i_1}}^{-1} \|)^{|I|} \quad \text{for } |I| \in \mathbb{Z}^+,$$

where $\pi_I^{(1)} : B_1 \rightarrow \mathcal{C}$ is the coordinate projection on a Banach algebra of power series B_1 in \mathcal{F}_k . Define

$$\delta_1 := \min_i (\| 2R_{\bar{t}_{i_1}}^{-1} \|^{-1}) \leq \liminf_{|I|} \| \pi_I^{(1)} \|^{-1/|I|}.$$

Now, following the arguments given in Lemma 3.1, if Ω_1 is the closed poly-disc centered at zero and radius $\delta_1/2$, then the mapping $\vartheta_1 : \bar{x}_1 \mapsto \sum_{i=0}^{\infty} \pi_I^{(1)}(\bar{x}_1) z^i$

of B_1 into $\text{Hol}(\Omega_1)$, a standard poly-disc algebra, is an algebraic isomorphism, continuous by the closed graph theorem. Since $\text{Hol}(\Omega_1)$ is semisimple, the same holds for B_1 . Clearly, the mapping $Q_1 : \bar{x} \mapsto \bar{x}_1$ of B into B_1 is also a continuous, injective homomorphism. This shows that B is a semisimple Fréchet algebra of power series in \mathcal{F}_k . This proves (i).

To prove (ii), one can follow the same arguments given in (ii) of [17] by noticing that $|\lambda|$ should be replaced by $\|\lambda\|_{\mathcal{C}^k}$, where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$.

Also, to prove (iii), one can follow the same arguments given in (iii) of Lemma 3.1. \square

4 Some Remarks on the Main Theorem and open questions

In 4.1 - 4.6 of [16] (which deals with the case $k = 1$), we give remarks on the hypotheses of the main theorem, with counterexamples, showing that the assumptions, considered on ϕ , t and (p_m) , cannot be dropped; nevertheless, as far as they go, these remarks also support the case of several-variable by considering the several-variable analogues of those counterexamples. In particular, if ϕ were isolated, then (ii) (of the main theorem and of Lemma 3.1)

is clearly impossible. Also, the remarks regarding: (i) a weaker hypothesis on \bar{t}_i in order to obtain a stronger form of the main theorem (cf. 4.4 of [16]), (ii) the independency of the choice of (p_m) (cf. 4.5 of [16]), and (iii) whether the sufficient condition on t_i is necessary for the existence of analytic variety at ϕ (cf. 4.6 of [16]), are of great interest. We, here, deal with a different set of remarks, specifically pertaining to the several-variable case.

4.1. As a special case of the main theorem, we characterize locally Stein algebras in the following corollary, whose proof we omit (see [10, Theorems 11.1.4 and 2.3.4] for details; proved for commutative uniform Fréchet algebras), and which also includes a copy of the result for the case of principal ideals, Riemann surface and locally Riemann algebras [16, Corollary 4.1].

Corollary 4.1 *Let A be a Fréchet algebra, with its topology defined by a sequence (p_m) of norms and with the Arens-Michael isomorphism $A = \varprojlim (A_m; d_m)$. Suppose that $M(A)$ contains an open subspace Y such that every closed maximal ideal M in Y is finitely generated such that: (a) $\dim(\overline{M^n}/\overline{M^{n+1}}) = C_{n+k-1,n}$ for all n ; and (b) the generators t_i , $i = 1, 2, \dots, k$, have the property that t_{i_m} is not a topological divisor of zero in A_m for all sufficiently large m . Then Y can be given the structure of a (reduced) Stein space in such a way that, for each $x \in A$, the restriction of \hat{x}*

to Y is analytic. In particular, if Y is locally compact and connected, such that the above conditions hold, then A is a locally Stein algebra (that is, the completion of $\hat{A}|Y$ with respect to the compact open topology is topologically and algebraically isomorphic to $\text{Hol}(Y)$). Conversely, if A is a locally Stein algebra, then every closed maximal ideal M (corresponding to a point in a (reduced) Stein space Y) is algebraically finitely generated.

We note that Banach algebras satisfying Lemma 3.1 (in particular, the poly-disc algebra $A(D^k)$ and the algebra $H^\infty(U)$ of all bounded analytic functions on some bounded domain U of \mathcal{C}^k , which are not nuclear), $A^\infty(\Gamma^k)$ (which is nuclear) and Stein algebras are locally Stein algebras whereas $C^\infty(\mathbb{R}^k)$ is not a locally Stein algebra. Moreover, $A = \{f \in \text{Hol}(\mathcal{C}) : f(0) = f(i) \text{ for all } i = 1, 2, \dots\}$ is a uniform Fréchet algebra, being a closed subalgebra of the Stein algebra $\text{Hol}(\mathcal{C})$; it is shown in 11.1.5 of [10] that A is not Stein, but it is, indeed, locally Stein. Hence this example poses an obvious question: is every closed, unital subalgebra of a locally Stein algebra a locally Stein algebra? Similarly, it is of interest to get a criterion when a quotient of a locally Stein algebra is itself locally Stein. In the literature, there are examples of complex function algebras with no analytic structure in their spectra, but in the case $k = 2$ (see [10, Remark, p. 235] for more

references).

4.2. It is easy to see that if the hypothesis (2) of Theorem 2.5 of [18] holds, then $\dim(\overline{M^n}/\overline{M^{n+1}}) = C_{n+k-1,n}$ for all n , by the remarks following Theorem 2.3 of [18]; the Beurling-Fréchet algebras $\ell^1(\mathbb{Z}^k, \Omega)$ of semi-weight type show that the converse is not true. Thus, in a special case, Lemma 3.1 is a generalization of Read's Theorem. In fact, the latter part of the hypothesis (2) of Theorem 2.5 of [18] is a topological assumption whereas the hypotheses of Lemma 3.1 are of purely algebraic nature. In addition, by the Hilbert-Serre theorem (see [20, p. 232]), there is a polynomial P of degree exactly $k - 1$ such that $P(n) = \dim(\overline{M^n}/\overline{M^{n+1}}) = C_{n+k-1,n}$ for all large n , and k is the dimension at the origin of the variety ϕ .

4.3. In Theorem 4.1 of [18], Read assumed a strictly weaker hypothesis of assuming $\dim(M/M^2)$ is finite instead a maximal ideal M being algebraically finitely generated, giving of the generalization of the Gleason's result in the Banach algebra case ([18, Example 5.1] shows that varieties thus obtained need not be, in general, open in the Gel'fand topology, and [18, Example 5.3] shows that such sufficient conditions are far from being necessary, even for open analytic structure; see 4.4 below). One naturally conjectures that the main theorem and Corollary 4.1 also hold true with this strictly weaker

hypothesis.

4.4. To see an application of our corollary, we recall a problem which in the literature is known as the Gleason problem [9]: to decide whether the maximal ideal in A , where A is either $H^\infty(\Omega)$ or $A(\Omega)$, Ω a bounded domain in \mathcal{C}^k , consisting of functions vanishing at the origin is algebraically finitely generated by the coordinate functions (see [3] and other references therein for more details). Further, we recall that a domain Ω has the Gleason A -property if the problem has an affirmative solution at all points of Ω . It is meaningful to pose this problem for locally Stein algebras in an appropriate manner: to decide whether the maximal ideal (corresponding to a point $p \in X \subset M(A)$) in A , where A is a locally Stein algebra, consisting of the Gel'fand transforms of elements of A vanishing at a point $p \in X \subset M(A)$, is algebraically finitely generated. This was actually the problem posed by Gleason in [9], and he mentioned that if the maximal ideal corresponding to the origin is algebraically finitely generated, then it is finitely generated by the coordinate functions by Theorem 2.2 (which would obviously hold true in the Fréchet algebra case); see [9, p. 131-132] for more details. We say that a subspace X of $M(A)$ has the Gleason A -property if the problem has an affirmative solution at all points of X .

The above corollary says that the subspace Y has the Gleason A -property, where A is a locally Stein algebra; in particular, A can be either $H^\infty(\Omega)$ or $A(\Omega)$, Ω a bounded domain in \mathcal{C}^k containing the origin. Thus we have given an abstract touch to the affirmative solution of the Gleason problem, and so the abstract method given here recaptures the classical results obtained by Fornæss and Øvrelid [8], Kerzman and Nagel [11], and Backlund and Fällström (see [3] and other references therein for a list of papers on the Gleason problem), that is, a bounded domain Ω in \mathcal{C}^k has the Gleason A -property, where A is either $H^\infty(\Omega)$ or $A(\Omega)$ and Ω a (strictly or weakly) pseudoconvex domain in \mathcal{C}^k with various boundary conditions.

Next, [8] is a good reference; in the final paragraph of §1, the authors state the main theorem can still be proved by replacing $A(\Omega)$ by various holomorphic Hölder- and Lipschitz-spaces and by replacing the coordinate functions as the generators by arbitrary generators of the maximal ideal in these spaces. They mention a future paper, but, as far as we know, it has never been published. In this connection, we can consider the above application of our corollary, which establishes the similar claim, but for locally Stein algebras.

4.5 We say that A satisfies the weak identity theorem in $\phi \in M(A)$ if there is a (fixed) neighbourhood U of ϕ in $M(A)$ such that $\hat{x}|_U \equiv 0$ for each $\hat{x} \in \hat{A}$ which vanishes in an arbitrary neighbourhood of ϕ . It follows from the theory of several complex variables that A satisfies the weak identity theorem in ϕ if there is an analytic variety at ϕ . Thus, the statement (iii) of the main theorem (and Lemma 3.1, too) shows that the weak identity theorem holds for A .

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